**Determinants**

Table of Contents

[Properties of Determinants 2](#_Toc66220660)

[Formulae for the Determinant 9](#_Toc66220661)

[The Pivot Formula 9](#_Toc66220662)

[The Big Formula 9](#_Toc66220663)

[Determinant by Co-Factor 12](#_Toc66220664)

[False Expansion Theorem 16](#_Toc66220665)

[Inverse by Co-Factor 17](#_Toc66220666)

[Cramer’s Rule 20](#_Toc66220667)

## Properties of Determinants

A determinant is a number associated with every square matrix. For a matrix, , it is often written as or . There are two things we already know about the determinant. When the determinant is not , the matrix is invertible. When the determinant is , the matrix is singular.

We shall start learning about determinants by going over a few properties of determinants. The first three are the absolute truths of determinants that we need to accept. Every other property can be derived from these.

The first property is that the determinant of the identity matrix is .

The second property looks at what happens if we exchange two rows of the matrix. In this case, the sign of the determinant is reversed.

From just these two properties, we now know the determinant of every permutation matrix in the world.

If the number of row exchanges was even , and if the number of row exchanges was odd, .

Thus,

and

We also know that the general determinant for a matrix, . We can use this to check that the properties are indeed correct, but in reality, the properties will give us the general formula for the determinant of an matrix.

The third property is about linear combinations and can be subdivided into two parts.

The first part of says that if we multiple one of the rows of a matrix by a number , then the determinant of the new matrix is the times the determinant of the old matrix, i.e. .

The second part says that .

The determinant is a linear function of the first row, if all the other rows stay the same. This is also true for each of the other rows separately. Note that this does **not** mean that .

The fourth property states that if two rows are equal, then the determinant will be . Suppose two rows in a matrix are equal. We can easily prove that the determinant is . If we exchange the two rows, we will get the same matrix and thus the determinant should be the same. However, property 2 tells us that the sign changes when we exchange two rows like this. The only value that is the same regardless of whether it is positive or negative is . We also know that the matrix cannot be invertible if it has two equal rows, and thus must have a determinant of .

The fifth property has to do with elimination matrices. It states that when we perform some elimination like ‘subtract from ’, the determinant remains unchanged, i.e. .

Consider a matrix. . After elimination, .

This also derives from the third and fourth properties. The second part of the third property tells us that a linear combination in one of the rows can be broken into two parts, so: . The first part of the third property tells us that if one of the rows is multiplied by a constant, the determinant of the new matrix is the determinant of the old matrix times the constant, so: . The fourth property tells us that the determinant of a matrix with two equal rows is , so: .

The sixth property states that a row of s leads to the determinant being . The first part of the third property can help us prove this. Say we have a matrix . If we multiply the first row by , then . The only way this can be true is if the determinant is .

Property seven deals with triangular matrices. Say we have a matrix . The determinant for an upper triangular matrix like this is the product of the pivots, i.e. .

However, this value is only applicable if there were no row exchanges in the path to getting to the upper triangular matrix. If we had to perform any row exchanges, we would need to change the sign of the result accordingly.

We can prove this property by performing upwards elimination on . By doing this, we will have only the diagonal left in the matrix and every other value will be (assuming the diagonal itself does not contain any s). This makes no difference since we know that the determinant is unaffected by elimination from the fifth property. Now we can use the first part of the third property. If we take out each of the pivots one by one, we will be left with a diagonal that consists only of s, which is the identity matrix. We know that the identity matrix has a determinant of , from property 1, and since we are multiplying this with all the pivot values, the determinant of the original matrix must be the product of the pivot values.

We also need to consider the case where some pivot value is . After performing upwards elimination, we would have a row of s in that case. From property six, we know that this would give us a determinant of . The product of the diagonals would also be in this case.

The eighth property tells us that the determinant of a matrix is , when is a singular matrix and that the determinant is not if is invertible. This is easily proven since if we perform elimination of a singular matrix, we will get a row of s and property six tells us that in that case, the determinant must be . This is also essentially the last part of the seventh property, where we saw what happens if a pivot value is , which happens with singular matrices.

We can now look at how we got the formula. If we perform elimination on the matrix , we get . Thus, since property seven tells us that the determinant of an upper triangular matrix is the product of its diagonals, the determinant here is . Of course, if , we need to perform an exchange first, and if that does not work the matrix must be singular. This knowledge is much more practical since we can use it to find the determinant of an matrix, simply by performing elimination.

The properties we have seen so far have dealt with elimination and pivot values. The next two are different.

Property nine states that the determinant of the product of two matrices the product of the determinants of the two matrices individually, i.e. . This is a very useful property that can help us quickly find the determinant of a lot of matrices.

For example, we can find the determinant of . We know that and we know that . Thus, if we can find the determinant of , . If we have a diagonal matrix , its determinant is and its inverse is , which has a determinant of so our conclusion checks out. It also makes sense for the case where is singular, since then which leaves us with an undefined determinant for , which makes sense since doesn’t even exist.

We also know that the determinant of is . The determinant of however, is **not** . Instead we must multiply by for every row, which leaves us with .

The tenth and final property is that . For the matrix , the transpose is , which has the same determinant .

This property tells us that every property we have seen applied to rows also applies to columns. If a column is all s, the determinant is also , since we can just transpose it to have a row of all s which we know has a determinant of from property six. If we exchange two columns, it reverses the sign of the determinant just like we saw it do for rows in property two.

For the proof of the tenth property,

(from property nine)

The determinant of a lower triangular matrix is , since we can get rid of all the values in the lower part. The same is true for . Thus,

There is one problem that needs to be discussed without which all the properties fall apart. What if we could form the same matrix by performing 7 row exchanges and 10 row exchanges. According to the second property, we should have a negated and a non-negated determinant respectively. The thing to understand here is that this simply is not possible. Two permutations can result in the same matrix, but that only works in sets of odd or even numbers, meaning if we get a matrix with an odd number of permutations, we may get the same matrix with another odd number of permutations, but never with an even number of permutations.

|  |  |
| --- | --- |
| Property 1 |  |
| Property 2 | Exchanging 2 rows flips the sign of the determinant. |
| Property 3 |  |
|  |
| Property 4 | If two rows in a matrix are equal, the determinant is . |
| Property 5 |  |
| Property 6 | If a matrix has a row of s, the determinant is . |
| Property 7 |  |
| Property 8 | if is singular and if is invertible |
| Property 9 |  |
| Property 10 |  |

## Formulae for the Determinant

### The Pivot Formula

The pivot formula is the most commonly used method to find the determinant of any matrix. Given a matrix , we perform elimination to divide it into two parts and , the lower triangular matrix and the upper triangular matrix respectively. We know from the ninth property we learnt in the previous lecture. The lower triangular matrix has s along its pivot so its determinant is . The determinant of is the product of every value in its diagonal. Thus, the determinant of is the product of the values on the diagonal of the upper triangular matrix. Of course, if we perform any row exchanges during the elimination process, the sign of the determinant may change.

### The Big Formula

Consider the matrix . From the third property from the previous lecture, we know that we can write

We can further split this into

The first and fourth terms have a column of s which means it’s a singular matrix.

We know from the tenth property that rules that apply to rows also apply to columns and we know from the sixth property that matrices with a row of s have a determinant of . Thus, both the first and fourth terms have a determinant.

From the seventh property, we know that the determinant of a diagonal matrix is the product of the cells on its diagonal, so the determinant of the second term is .

If we flip the rows of the third term and apply the second and seventh properties, we can also know that the determinant of the third term is .

Thus, .

We can apply the exact same method to any square matrices. First, we keep the second and third rows same and split the first row into three parts. Then we split the second row for each of those parts into three parts. Finally, we split the third row for all nine parts into three parts each to give a total of twenty-seven parts. We remove the parts that have columns, flip rows where needed and multiply diagonals to get the answer.

This is obviously quite a long and tiring process, so let us try and shorten it. First, we can forget about the columns and only look at the valid parts. For the matrix

the only valid parts are going to be

, , , , and

Notice that each of the valid terms must have one member in each row and column. Knowing this, it is easy to figure out which one the valid terms should be without actually writing down all the terms first. We can then switch rows appropriately and carry on with calculations. From the above, the determinant would be

But even this is not enough. We now need to make a jump to the general formula for an matrix. We got terms for a matrix, terms for a matrix and we would get terms for a matrix. Thus, for an matrix, we will get terms. This is because, the member in the first row can be chosen in ways, the member in the second row ways and so on. This also means that this method is extremely inefficient.

For an matrix, the determinant will follow this formula:

where the set of column numbers is some permutation of .

This formula can be used to prove that all of the properties we have learn are true. Let’s look at an example to make the use of this formula a little clearer.

Consider . Consider the number of ways we can get a valid term from this. The terms is a valid term, and it requires two permutations to get in the correct order so its sign should be positive. is another term, and this requires just one exchange so its sign is negative. There are no other possible valid terms. Thus, the determinant we get is . This is a singular matrix. We could of course have proven this in other ways such as by finding the null space.

### Determinant by Co-Factor

Consider the matrix and pick any of the rows. For now, let’s say we picked the first row. Now one by one take each of the members in the first row and ignore the rest of the row and column that it is on. The cells that remain are called the co-factor for the cell that we picked. For the cells , and we have:

The co-factors are labelled , and respectively. Since the co-factors are simple matrices, we can find the determinants for them quite easily. For , the determinant is , for it is and for it is . The determinants are called the minor of the element at the index, i.e. is the minor of , is the minor of and so on. There is another part to the co-factor, and that is the sign. The sign for any co-factor is given by . Thus,

Co-Factor

The determinant of the original matrix is given by:

A similar formula would be true had we picked either of the other two rows to begin with. In general, for an matrix, the determinant by co-factor is given by the formula:

Example 1:

Example 2:

Notice that there is a very large distance between and in the first column. Due to the way this matrix is setup, transposing the matrix will give us co-factors that are very easy to solve.

If we take the first row and use co-factors to find the determinant:

Notice that the co-factor of is a lower triangular matrix with a determinant that is the product of the cells in its diagonal and the co-factor of is an upper triangular matrix with a determinant that is the product of the cells in its diagonal. We transposed the original matrix in order to get ourselves into this easy situation. Thus,

### False Expansion Theorem

The false expansion theorem states that if is a matrix then given that .

Essentially, this means that if we try to find the determinant of a matrix by multiplying a row with co-factors that are for a different row, then the result will be .

Consider that . Using co-factors, we know that

Now say we tried to use to use the second row instead, but still took the co-factors for the first row.

The False Expansion Theorem can be proven as follows. Say we have two matrices and such that is identical to except that the th row of is the same as its th row. When a matrix has two identical rows, we know that its determinant must be .

Let’s say instead we want to find the determinant of using the co-factor method, using its th row. Thus

Notice that if we try to find the determinant of in the same method using the th row of , we will obtain the exact same co-factors. The only difference would be that instead of multiplying by the th row, we would be multiplying by the th row. Thus, the only difference between the formulae for and is that in , we are multiplying the co-factors by the incorrect th row. Thus, if we try to find the determinant of a matrix using the co-factor method and we multiply the co-factor of one row with another row’s elements, we will get as the result.

### Inverse by Co-Factor

We essentially know that

where is the co-factor matrix of . A co-factor matrix is simply the matrix containing the corresponding co-factors of every element in in their respective positions.

Since we are multiplying every value of the th row of with its corresponding co-factor, it would be easier to consider the two rows as vectors, which would give us the same result with the dot products of those vectors.

Here, the two rows have been transposed so that we can say they are vectors.

As always,

[]

We can re-write these in this way as well

where we are essentially saying that the th row of the matrix is being multiplied by the transpose of the th row of the co-factor matrix for . This is still the same thing as the original long equation.

And obviously

[]

Now consider the matrix . The co-factor matrix for , . Thus,

Essentially, for , we will get for every cell along the diagonal and in every other cell. Another way of looking at this is that the determinant is multiplied by the identity matrix.

If is invertible, its determinant is not and it has an inverse. Thus, we can write

Example:

## Cramer’s Rule

We know that , where is the co-factor matrix of . From this, if we are given an equation , we can easily solve for .

If we actually try to solve this, we will see that is made up of multiple components, , and so on since . Each of these components are the result of the respective multiplication of one row of the matrix and , i.e. , . Of course, this leads to the conclusion that each of these multiplications is giving us the determinant of some unknown matrix, since each follows the pattern of the co-factor method of finding the determinant.

Say . Cramer figured out that this matrix is just the matrix , with its first column replaced with . Notice what this actually looks like. If we try to find the determinant of , using co-factors along the first column, we will find that . This proves the theory. Thus, .

Both Cramer’s rule and the equation to find can be useful in situations where we are having to deal with algebra, but they are actually terrible methods to follow for matrices themselves because of the number of computations involved. A software like MATLAB would use the far easier elimination method we learnt at the beginning of the course.